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Certified quantum non-demolition measurement of material systems

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Abstract. An extensive debate on quantum non-demolition (QND) measurement, reviewed in Grangier *et al* (1998 *Nature* **396** 537), finds that true QND measurements must have both non-classical state-preparation capability and non-classical information-damage tradeoff. Existing figures of merit for these non-classicality criteria require direct measurement of the signal variable and are thus difficult to apply to optically-probed material systems. Here we describe a method to demonstrate both criteria without need for to direct signal measurements. Using a covariance matrix formalism and a general noise model, we compute meter observables for QND measurement triples, which suffice to compute all QND figures of merit. The result will allow certified QND measurement of atomic spin ensembles using existing techniques.

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Contents

1. Introduction

A quantum non-demolition (QND) measurement is one which provides information about a quantum variable while leaving it unchanged and accessible for future measurements. The approach was originally suggested as a means to avoid measurement back-action in gravitational wave detection [1–5]. QND measurements of optical fields both provided the first demonstrations and led to a considerable refinement of the understanding of QND measurements in practice [6]. More recently, QND measurements have been employed to prepare spin-squeezed atomic states [7–11] and with nano-mechanical systems [12].

In a generic QND measurement, a 'meter' and a 'system' variable interact via a selected Hamiltonian. The meter can then be directly measured to gain indirect information about the system. In the context of continuous-variable optical QND measurements, the question of when a measurement should be considered QND has been much discussed (see [6] and references therein). Two distinct non-classicality criteria emerge: A state preparation criterion requires small uncertainty in the system variable after the measurement while a second criterion describes the information-damage tradeoff in the measurement. While some operations such as filtering or optimal cloning can be non-classical in one or the other criterion, a true QND measurement is non-classical in both [6]. Similar criteria have been developed for discrete-variable systems such as qubits [13], but are outside the scope of this article.

With the aid of figures of merit [14–16] describing the quantum-classical boundary, optical QND measurements satisfying both criteria have been demonstrated [15, 17–25]. These figures of merit make use of the fact that the optical signal beam, after the QND measurement, can be verified by a direct, i.e. destructive, measurement with quantum-noise-limited sensitivity. Direct measurement of the system variable is typically not available in atomic QND. Instead, repeated QND measurement has been used to show the state preparation criterion [8–11, 26] by conditional variance measurements. Here we show how repeated QND measurements can also be used to test the information-damage tradeoff, and thus to certify full QND performance without direct access to the system variable.

2. Model

As in the pioneering work by Kuzmich *et al* [7, 27], we consider the collective spin of an atomic ensemble, described by the vector angular momentum operator **J**. We note that a variety of other physical situations are described in the same way, e.g. by using a pseudo-spin to describe a clock transition [9]. The optical polarization of any probe pulse is described by a vector Stokes operator **S**

$$S_i \equiv \tfrac{1}{2}\mathbf{a}^\dagger \sigma_i \mathbf{a}, \tag{1}$$

$i = x, y, z$ where σ_i are the Pauli matrices, $\mathbf{a} \equiv \{a_+, a_-\}^{\mathrm{T}}$ and $a_\pm$ are annihilation operators for circular-plus and circular-minus polarizations.

We define Stokes operators **P**, **Q** for the first and second pulses, respectively. The operators **J**, **P**, **Q** each obey the angular momentum commutation relation $[L_x, L_y] = \mathrm{i}\, L_z$ and cyclic permutations (for simplicity, we take $\hbar = 1$). For notational convenience, we define the combined optical variables $\mathbf{C} \equiv \mathbf{P} \oplus \mathbf{Q}$ and the total variable $\mathbf{T} \equiv \mathbf{J} \oplus \mathbf{C}$. We will be interested in the average values of these operators, which we write as $\bar{\mathbf{J}} \equiv \langle \mathbf{J} \rangle$ and similar, and the covariance matrices, which we write as

$$\tilde{J} \equiv \tfrac{1}{2}\langle \mathbf{J} \wedge \mathbf{J} + (\mathbf{J} \wedge \mathbf{J})^{\mathrm{T}} \rangle - \langle \mathbf{J} \rangle \wedge \langle \mathbf{J} \rangle \tag{2}$$

and similar. We assume Gaussian states with amplitudes $\gg 1$. In this scenario the averages and covariances fully describe the system. Our approach follows Madsen and Mølmer [28] but treats all spin and polarization components, as developed in [29, 30].

We assume that the input probe pulses are polarized as $\bar{P}_x^{(\mathrm{in})} = \bar{Q}_x^{(\mathrm{in})} = \bar{S}_x^{(\mathrm{in})}$ and that the other average components are zero. Similarly, $\bar{J}_x^{(\mathrm{in})} = |J|$ while other average components are zero. We take the initial covariance matrix for the system to be

$$\tilde{T}_0 = \tilde{J} \oplus \tilde{C}. \tag{3}$$

This form of the covariance matrix allows for arbitrary prior correlations (including correlated technical noise) among the two optical pulses, but no prior correlations between the atoms and either optical pulse.

The interaction is described by an effective Hamiltonian

$$H_{\mathrm{eff}} = g J_z S_z, \tag{4}$$

where g is a constant [31].

This QND interaction, to lowest order in $g\tau$, where τ is the interaction time of the pulse and atoms, produces a rotation of the state, $\mathbf{T}^{(\mathrm{out})} = \mathbf{T}^{(\mathrm{in})} - \mathrm{i}\tau[\mathbf{T}^{(\mathrm{in})}, H_{\mathrm{eff}}]$. This has the effect of imprinting information about J_z on the light without changing J_z itself:

$$S_y^{(\mathrm{out})} = S_y^{(\mathrm{in})} + \kappa' S_x^{(\mathrm{in})} J_z^{(\mathrm{in})}, \tag{5}$$

$$J_y^{(\mathrm{out})} = J_y^{(\mathrm{in})} + \kappa' J_x^{(\mathrm{in})} S_z^{(\mathrm{in})} \tag{6}$$

and

$$\mathcal{O}^{(\mathrm{out})} = \mathcal{O}^{(\mathrm{in})} \tag{7}$$

for any variable $\mathcal{O} \notin \{S_y, J_y\}$, i.e. including J_z. Here $\kappa' = g\tau$ and **S** is **P** or **Q** depending on which pulse–atom interaction is being described. The rotation can be described by a linear transformation

$$\mathbf{T}^{(\mathrm{out})} = M_P \mathbf{T}^{(\mathrm{in})} \tag{8}$$

and thus

$$\tilde{T}^{(\text{out})} = M_P \tilde{T}^{(\text{in})} M_P^{\text{T}}, \tag{9}$$

where M_P is equal to the identity matrix, apart from the elements $(M_P)_{2,6} = \kappa' \bar{J}_x^{(\text{in})}$, and $(M_P)_{5,3} = \kappa' \bar{S}_x^{(\text{in})}$. For later convenience, we define $\kappa \equiv \kappa' \bar{S}_x^{(\text{in})} = g\tau \bar{S}_x^{(\text{in})}$.

The effect of the second pulse is described by the matrix $M_Q = X_{PQ} M_P X_{PQ}$ where

$$X_{PQ} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \otimes I_3 \tag{10}$$

exchanges the roles of P and Q, and I_3 is the 3×3 identity matrix.

3. Reduction of uncertainty by quantum non-demolition (QND) measurement

We first consider the case in which the interaction does not introduce additional noise (although both the input atomic and optical states may be noisy). After interaction with the first pulse, but before the arrival of the second pulse, the state is described by $\tilde{T}_P \equiv M_P \tilde{T}_0 M_P^{\text{T}}$. A component P_y of the first pulse is measured. Formally, this corresponds to projection along the axis $\mathbf{m}_P \equiv \{0, 0, 0, 0, 1, 0, 0, 0, 0\}^{\text{T}}$, and $\tilde{T}_P$ is reduced to

$$\tilde{T}_{PD} = \tilde{T}_P - \tilde{T}_P (\Pi_Q \tilde{T}_P \Pi_Q)^{\text{MP}} \tilde{T}_P^{\text{T}} = \tilde{T}_P - \tilde{T}_P \Pi_Q \tilde{T}_P^{\text{T}} / \text{Tr}[\Pi_Q \tilde{T}_P], \tag{11}$$

where $\Pi_Q \equiv \mathbf{m}_P \wedge \mathbf{m}_P$ is the projector describing the measurement and $()^{\text{MP}}$ indicates the Moore–Penrose pseudo-inverse.

We can directly calculate the resulting variance of J_z,

$$E[\text{var}(J_z)|P_y] \equiv (\tilde{T}_{PD})_{3,3} = \tilde{J}_{3,3} \frac{\tilde{C}_{2,2}}{\kappa^2 \tilde{J}_{3,3} + \tilde{C}_{2,2}}. \tag{12}$$

This has a natural interpretation: the variance of the detected projection P_y has two contributions: $\kappa^2 \tilde{J}_{3,3}$ from the atomic signal and $\tilde{C}_{2,2}$ from the pre-existing optical noise. $\tilde{J}_{3,3}$ is reduced by the factor $1/(1 + \text{SNR})$ where SNR is the signal-to-noise ratio of the measurement. A similar result is found in [28]. This post-measurement variance of the signal variable describes the state-preparation capability of the QND measurement. Absent the ability to directly measure J_z, we must look for observables which contain this same information.

4. Observable correlations

After interaction with both the first and second pulses, we have $\tilde{T}_{PQ} \equiv M_Q \tilde{T}_P M_Q^{\text{T}}$. This matrix contains the variances and correlations that are directly measurable, namely those of the two light pulses. These are

$$\text{var}(P_y) = \tilde{C}_{2,2} + \kappa^2 \tilde{J}_{3,3}, \tag{13}$$

$$\text{var}(Q_y) = \tilde{C}_{5,5} + \kappa^2 \tilde{J}_{3,3}, \tag{14}$$

$$\text{cov}(P_y, Q_y) = \tilde{C}_{2,5} + \kappa^2 \tilde{J}_{3,3}. \tag{15}$$

We note that for $\kappa = 0$, e.g. if the atoms are removed, the values are

$$\mathrm{var}_{\mathrm{NA}}(P_y) = \tilde{C}_{2,2}, \tag{16}$$

$$\mathrm{var}_{\mathrm{NA}}(Q_y) = \tilde{C}_{5,5}, \tag{17}$$

$$\mathrm{cov}_{\mathrm{NA}}(P_y, Q_y) = \tilde{C}_{2,5}. \tag{18}$$

We see that the state preparation capability can be expressed in terms of measurable quantities as

$$E[\mathrm{var}(J_z)|P_y] = \tilde{J}_{3,3}\frac{\mathrm{var}_{\mathrm{NA}}(P_y)}{\mathrm{var}(P_y)}, \tag{19}$$

which uses the variance of the two measurements to determine the SNR. Another formulation,

$$E[\mathrm{var}(J_z)|P_y] = \tilde{J}_{3,3}\frac{\mathrm{var}_{\mathrm{NA}}(P_y)}{\mathrm{var}_{\mathrm{NA}}(P_y) + \mathrm{cov}(P_y, Q_y) - \mathrm{cov}_{\mathrm{NA}}(P_y, Q_y)} \tag{20}$$

expresses the residual variance in terms of the atomic contribution to the correlation between first and second pulses.

These simple expressions are only valid for noise-free interactions, however. In a real experiment, other effects are present which introduce both noise and losses in the atomic and optical variables. We now account for these other effects.

5. General noise and loss

We now consider noise produced in the atom-light interaction itself, as well as losses. The noise model we employ is very general. The interaction of the first pulse with the atoms is described by (see appendix)

$$\tilde{T}_P = M_P\tilde{T}_0 M_P^{\mathrm{T}} + N_P, \tag{21}$$

where N_P is a matrix describing noise in the first pulse. We assume that the coherent part of the interaction is $M_P \equiv r_A I_3 \oplus r_L I_3 \oplus I_3$ apart from the elements $(M_P)_{2,6} = \kappa \bar{J}_x^{(\mathrm{in})}$, and $(M_P)_{5,3} = -\kappa \bar{S}_x^{(\mathrm{in})}$.

Here r_A, r_L describe the fraction of atoms and photons, respectively, that remain after the interaction. Thus M_P includes both the effect of H_{eff} and linear losses. We leave N_P completely general, except that it does not affect $\mathbf{Q}$: $N_P \equiv N \oplus 0I_3$, where N is a six-by-six symmetric matrix.

Similarly, we describe interaction with the second pulse as

$$\tilde{T}_{PQ} = M_Q\tilde{T}_P M_Q^{\mathrm{T}} + N_Q, \tag{22}$$

where $M_Q = X_{PQ}M_P X_{PQ}$ and $N_Q = X_{PQ}N_P X_{PQ}$.

Note that we assume that both the interaction M and the noise N are the same for the first and second pulses (but act on different variables, naturally). This implies that optical characteristics of the pulses such as detuning from resonance are the same, a condition that can be achieved in experiments. It also assumes that the noise generated by the interaction is incoherent and state-independent, as opposed to a more general, state-dependent noise $N(\mathbf{J}, \mathbf{S})$. Nevertheless, in many situations $\mathbf{J}$ and $\mathbf{S}$ are nearly constant (only small quantum components change appreciably), so that any reasonable $N(\mathbf{J}, \mathbf{S})$ would be effectively constant.

As above, we can directly calculate $\tilde{T}_{PD}$ and $\tilde{T}_{PQ}$ to find

$$E[\text{var}(J_z)|P_y] = \tilde{J}_{3,3} r_A^2 + N_{3,3} - \frac{(\kappa r_A \tilde{J}_{3,3} + N_{3,5})^2}{\kappa^2 \tilde{J}_{3,3} + r_L^2 \tilde{C}_{2,2} + N_{5,5}} \quad (23)$$

and

$$\text{var}(P_y) = r_L^2 \tilde{C}_{2,2} + \kappa^2 \tilde{J}_{3,3} + N_{5,5}, \quad (24)$$

$$\text{var}(Q_y) = r_L^2 \tilde{C}_{5,5} + \kappa^2 (r_A^2 \tilde{J}_{3,3} + N_{3,3}) + N_{5,5}, \quad (25)$$

$$\text{cov}(P_y, Q_y) = r_L^2 \tilde{C}_{2,5} + \kappa^2 (r_A \tilde{J}_{3,3} + N_{3,5}/\kappa). \quad (26)$$

Equation (16) still holds for the case with no atoms. We define

$$\delta\text{var}(P_y) \equiv \text{var}(P_y) - \text{var}_{\text{NA}}(P_y) r_L^2, \quad (27)$$

$$\delta\text{var}(Q_y) \equiv \text{var}(Q_y) - \text{var}_{\text{NA}}(Q_y) r_L^2, \quad (28)$$

$$\delta\text{cov}(P_y, Q_y) \equiv \text{cov}(P_y, Q_y) - \text{cov}_{\text{NA}}(P_y, Q_y) r_L^2, \quad (29)$$

where the r_L factors are included to account for atom-induced optical losses.

It is then simple to check that

$$E[\text{var}(J_z)|P_y] = \tilde{J}_{3,3} + \kappa^{-2} \left(\delta\text{var}(Q_y) - \delta\text{var}(P_y) - \frac{\delta\text{cov}^2(Q_y, P_y)}{\text{var}(P_y)} \right). \quad (30)$$

We note that the QND measurement reduces the variance of J_z if the quantity in parentheses is negative, i.e. if

$$\delta\text{cov}^2(Q_y, P_y) > \text{var}(P_y)[\delta\text{var}(Q_y) - \delta\text{var}(P_y)]. \quad (31)$$

Again, there is an intuitive explanation: $\delta\text{cov}(Q_y, P_y)$, which arises from the fact that both pulses measure the same atomic variable J_z, is a measure of the atom-light coupling. $[\delta\text{var}(Q_y) - \delta\,\text{var}(P_y)]$ expresses the difference in atom-induced noise between the first and second pulses. This difference indicates a change in the atomic state, namely an increase in $\text{var}(J_z)$. The condition of equation (31) compares these two effects and can be tested knowing the statistics of the various measurements on S_y and the optical transmission r_L. The factors κ^2, $\tilde{J}_{3,3}$ in equation (30) must be determined by independent means. For example, κ can be found by measuring the rotation of a state with known $\langle J_z \rangle \neq 0$ and $\tilde{J}_{3,3}$ from the number of atoms, or the observed noise scaling of a known state [32, 33].

6. Three-pulse experiments

The above description of two-pulse experiments can be extended straightforwardly to three or more pulses [29]. While a two-pulse experiment, plus prior knowledge of κ and $\tilde{J}_{3,3}$, gives sufficient information to find the post-measurement variance, and thus test the state-preparation property, a three-pulse experiment is required to find the other quantities used to characterize QND measurements.

If **R** denotes the Stokes vector of the third probe pulse, then statistics such as $\text{var}(R_y)$ and $\text{cov}(P_y, R_y)$ can be determined, and these in turn provide enough constraints to determine the loss and noise. Expanding our system to $\mathbf{T} \equiv \mathbf{J} \oplus \mathbf{P} \oplus \mathbf{Q} \oplus \mathbf{R}$, and defining interaction and noise operators M_R, N_R in the obvious way, a direct calculation finds several useful relations

$$r_A = \frac{\delta \text{cov}(P_y, R_y)}{\delta \text{cov}(P_y, Q_y)}, \tag{32}$$

$$r_A^2 = \frac{\delta \text{var}(R_y) - \delta \text{var}(Q_y)}{\delta \text{var}(Q_y) - \delta \text{var}(P_y)}, \tag{33}$$

$$\kappa^2 N_{3,3} = \delta \text{var}(Q_y) - \delta \text{var}(P_y) + \kappa^2 \tilde{J}_{3,3}\left(1 - r_A^2\right), \tag{34}$$

$$\kappa N_{3,5} = \delta \text{cov}(P_y, Q_y) - \kappa^2 \tilde{J}_{3,3} r_A, \tag{35}$$

$$N_{5,5} = \delta \text{var}(P_y) - \kappa^2 \tilde{J}_{3,3}. \tag{36}$$

7. Measures of QND performance

To quantify QND performance, Holland *et al* use the degree of correlation between various combinations of the input and output system variable $X = J_z$ and meter variable $Y = S_y$ variables [14]. They define three figures of merit, each of which is unity for an ideal QND measurement. These describe the measurement quality, the preservation of the initial value, and the state preparation capability, respectively:

$$C^2_{X^{\text{in}},Y^{\text{out}}} \equiv \frac{\text{cov}^2(X^{\text{in}}, Y^{\text{out}})}{\text{var}(X^{\text{in}})\text{var}(Y^{\text{out}})} = \frac{\kappa^2 \tilde{J}^2_{3,3}}{\tilde{J}_{3,3}(\tilde{T}_P)_{5,5}} = \frac{\kappa^2 \tilde{J}_{3,3}}{\text{var}(P_y)}, \tag{37}$$

$$\begin{aligned} C^2_{X^{\text{in}},X^{\text{out}}} &\equiv \frac{\text{cov}^2(X^{\text{in}}, X^{\text{out}})}{\text{var}(X^{\text{in}})\text{var}(X^{\text{out}})} = \frac{r_A^2 \tilde{J}^2_{3,3}}{\tilde{J}_{3,3}(\tilde{T}_P)_{3,3}}, \\ &= \frac{\kappa^2 \tilde{J}_{3,3} \delta \text{cov}^2(P_y, R_y)}{\delta \text{cov}^2(P_y, Q_y)[\delta \text{var}(Q_y) - \delta \text{var}(P_y) + \kappa^2 \tilde{J}_{3,3}]}, \end{aligned} \tag{38}$$

$$\begin{aligned} C^2_{X^{\text{out}},Y^{\text{out}}} &\equiv \frac{\text{cov}^2(X^{\text{out}}, Y^{\text{out}})}{\text{var}(X^{\text{out}})\text{var}(Y^{\text{out}})} = \frac{(\tilde{T}_P)^2_{3,5}}{(\tilde{T}_P)_{3,3}(\tilde{T}_P)_{5,5}}, \\ &= \frac{\delta \text{cov}^2(P_y, Q_y)}{\text{var}(P_y)[\delta \text{var}(Q_y) - \delta \text{var}(P_y) + \kappa^2 \tilde{J}_{3,3}]}. \end{aligned} \tag{39}$$

8. Non-classicality criteria

Roch *et al* [15] and Grangier *et al* [16] define non-classicality criteria using the conditional variance $\Delta X^2_{s|m}$, as in equation (30), and the quantities ΔX^2_m, the measurement noise referred to the input and ΔX^2_s, the excess noise introduced into the system variable. All are normalized by the intrinsic quantum noise of the system variable, a quantity which may depend on the system or the application. For example, in a spin-squeezing context the natural noise scale is $\tilde{J}_0 = |\langle J_x \rangle|/2 = \tilde{J}_{3,3}$, the J_z variance of the input x-polarized coherent spin state, i.e. the projection noise. Here we choose to normalize ΔX_m^2 by $\tilde{J}_0$, and $\Delta X_{s|m}^2$, ΔX_s^2 by $r_A \tilde{J}_0$, reflecting the reduction in size of the spin due to losses in the measurement process. The relation of information gained to damage caused is non-classical if $\Delta X_s \Delta X_m < 1$. We find

$$\Delta X_{s|m}^2 \equiv \frac{E[\mathrm{var}(J_z)|P_y]}{r_A \tilde{J}_0} = \frac{\delta\mathrm{cov}(P_y, Q_y)}{\delta\mathrm{cov}(P_y, R_y)} \left[1 + (\kappa^2 \tilde{J}_0)^{-1} \left(\delta\mathrm{var}(Q_y) - \delta\mathrm{var}(P_y) - \frac{\delta\mathrm{cov}^2(P_y, Q_y)}{\mathrm{var}(P_y)} \right) \right], \tag{40}$$

$$\Delta X_m^2 \equiv \frac{\tilde{C}_{2,2} r_L^2 + N_{5,5}}{\kappa^2 \tilde{J}_0} = \frac{\mathrm{var}(P_y) - \kappa^2 \tilde{J}_{3,3}}{\kappa^2 \tilde{J}_0}, \tag{41}$$

$$\Delta X_s^2 \equiv \frac{(\tilde{T}_P)_{3,3} - \tilde{J}_{3,3}}{r_A \tilde{J}_0} = \frac{\delta\mathrm{cov}(P_y, Q_y)[\delta\mathrm{var}(Q_y) - \delta\mathrm{var}(P_y)]}{\delta\mathrm{cov}(P_y, R_y)\kappa^2 \tilde{J}_0}. \tag{42}$$

9. Conclusions

Using the covariance matrix formalism and a general noise model, we have shown that full certification of QND measurements is possible without direct access to the system variable under study. We find that repeated probing of the same system gives statistical information sufficient to quantify both the state preparation capability and the information-damage tradeoff. The results enable certification of true QND measurement of material systems, and are directly applicable to ongoing experiments using QND measurements for quantum information [31] and quantum-enhanced metrology [33–35].

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Appendix

We derive equation (21) considering an extended pulse and using temporal sectioning, as described in [28, 29]. We divide the pulse duration into sections or time-slices $i = 1, \ldots, M_{\mathrm{sec}}$, with input and output Stokes operators $S_\alpha^{(\mathrm{in},i)}$, $S_\alpha^{(\mathrm{out},i)}$, respectively and $\alpha \in \{x, y, z\}$. We label the input and output atomic variables at the corresponding times $J_\alpha^{(\mathrm{in},i)}$, $J_\alpha^{(\mathrm{out},i)}$. By allowing M_{sec} to be large, we can avoid artifacts due to the discrete-time nature of the model. The variables appearing in equation (21) are the aggregate variables $S_\alpha^{(\mathrm{in})} \equiv \sum_{i=1}^{M_{\mathrm{sec}}} S_\alpha^{(\mathrm{in},i)}$, $S_\alpha^{(\mathrm{out})} \equiv \sum_{i=1}^{M_{\mathrm{sec}}} S_\alpha^{(\mathrm{out},i)}$ and $J_\alpha^{(\mathrm{in})} \equiv J_\alpha^{(\mathrm{in},1)}$, $J_\alpha^{(\mathrm{out})} \equiv J_\alpha^{(\mathrm{out},M_{\mathrm{sec}})}$.

Using equation (4) we find input-output relations, as in equation (5), but now we include noise operators $N_{\mathcal{O}}^{(i)}$ and amplitude decay via factors r_A, r_L which describe the remaining amplitude after a pulse. We write

$$S_y^{(\text{out},i)} = r_L(S_y^{(\text{in},i)} + \kappa'' S_x^{(\text{in},i)} J_z^{(\text{in},i)}) + N_{S_y}^{(i)}, \tag{A.1}$$

$$J_y^{(\text{out},i)} = r_A^{1/M_{\text{sec}}}(J_y^{(\text{in},i)} + \kappa'' J_x^{(\text{in},i)} S_z^{(\text{in},i)}) + N_{J_y}^{(i)} \tag{A.2}$$

and

$$\mathcal{O}^{(\text{out},i)} = (r_A^{1/M_{\text{sec}}}|r_L)\mathcal{O}_i^{(\text{in},i)} + N_{\mathcal{O}}^{(i)} \tag{A.3}$$

for all variables $\mathcal{O} \notin \{S_y, J_y\}$, i.e. including J_z, and where $(A|B)$ is A or B for atomic or optical variables, respectively. The atomic variable J_z accumulates noise as

$$J_z^{(\text{in},i+1)} = J_z^{(\text{out},i)} = r_A^{i/M_{\text{sec}}} J_z^{(\text{in})} + \sum_{j=1}^{i} r_A^{(i-j)/M_{\text{sec}}} N_{J_z}^{(j)}, \tag{A.4}$$

so that

$$S_y^{(\text{out},i)} = r_L \left[S_y^{(\text{in},i)} + \kappa'' S_x^{(\text{in},i)} \left(r_A^{(i-1)/M_{\text{sec}}} J_z^{(\text{in})} + \sum_{j=1}^{i-1} N_{J_z}^{(j)} \right) \right] + N_{S_y}^{(i)}. \tag{A.5}$$

Computing the aggregate variables we find

$$S_y^{(\text{out})} = r_L \left(S_y^{(\text{in})} + \kappa'' S_x^{(\text{in})} \xi J_z^{(\text{in})} + \kappa'' S_x^{(\text{in})} \sum_{i=1}^{M_{\text{sec}}} \frac{i-1}{M_{\text{sec}}} N_{J_z}^{(i)} \right) + \sum_{i=1}^{M_{\text{sec}}} N_{S_y}^{(i)}, \tag{A.6}$$

$$J_y^{(\text{out})} = r_A J_y^{(\text{in})} + \kappa'' J_x^{(\text{in})} \left(\sum_i r_A^{(i-1)/M_{\text{sec}}} S_z^{(\text{in},i)} \right) + \sum_{i=1}^{M_{\text{sec}}} N_{J_y}^{(i)}, \tag{A.7}$$

where $\xi = M_{\text{sec}}^{-1} \sum_{i=1}^{M_{\text{sec}}} r_A^{(i-1)/M_{\text{sec}}} = (r_A - 1)/[M_{\text{sec}}(r_A^{1/M_{\text{sec}}} - 1)]$, which approaches unity for $r_A \to 1$, and describes the weighted accumulation of signal. Also,

$$\mathcal{O}^{(\text{out})} = (r_A|r_L)\mathcal{O}_i^{(\text{in})} + \sum_{i=1}^{M_{\text{sec}}} N_{\mathcal{O}}^{(i)}, \tag{A.8}$$

for all other aggregate variables $\mathcal{O}$. By construction, each segment of pulse contains the same average number of photons, which are S_x-polarized. The $S_z^{(\text{in},i)}$ are thus independent zero-mean random variables of equal variance $\text{var}(S_z^{(\text{in},i)}) = \text{var}(S_z^{(\text{in})})/M_{\text{sec}}$, so that their weighted sum $\sum_{i=1}^{M_{\text{sec}}} r_A^{(i-1)/M_{\text{sec}}} S_z^{(\text{in},i)}$ is a random variable with zero mean and variance $\text{var}(S_z^{(\text{in})})\xi$. As such, the effect of non-unit r_A is to alter the apparent light–atom coupling. Defining $\kappa' \equiv \kappa''\xi$, we recover the coupling of equation (5). Also, defining N_P as the covariance matrix of the noise vector $\mathbf{n} \equiv \sum_{i=1}^{M_{\text{sec}}} (N_{J_x}^{(i)}, N_{J_y}^{(i)}, N_{J_z}^{(i)}, N_{S_x}^{(i)}, N_{S_y}^{(i)} + r_L \kappa'' S_x^{(\text{in})}(i-1) N_{J_y}^{(i)}/M_{\text{sec}}, N_{S_z}^{(i)})$, we arrive to equation (21).

Additional material. The calculations described in this article can be performed in *Mathematica* using the notebook 'ThreePulseCMCalculator,' available as an ancillary file at http://arxiv.org/abs/1203.6584.

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