Efficient Quantification of Non-Gaussian Spin Distributions

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We study theoretically and experimentally the quantification of non-Gaussian distributions via nondestructive measurements. Using the theory of cumulants, their unbiased estimators, and the uncertainties of these estimators, we describe a quantification which is simultaneously efficient, unbiased by measurement noise, and suitable for hypothesis tests, e.g., to detect nonclassical states. The theory is applied to cold ^{87}Rb spin ensembles prepared in non-Gaussian states by optical pumping and measured by nondestructive Faraday rotation probing. We find an optimal use of measurement resources under realistic conditions, e.g., in atomic ensemble quantum memories.

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Introduction.—Non-Gaussian states are an essential requirement for universal quantum computation [1,2] and several quantum communication tasks with continuous variables, including improving the fidelity of quantum teleportation [3] and entanglement distillation [4,5]. Optical non-Gaussian states have been demonstrated [6–10] and proposals in atomic systems [11–14] are being actively pursued. In photonic systems, histograms [15] and state tomography [6,7,9,10] have been used to show non-Gaussianity, but require a large number of measurements. For material systems with longer time-scales these approaches may be prohibitively expensive. Here we demonstrate the use of cumulants, global measures of distribution shape, to show non-Gaussianity in an atomic spin ensemble. Cumulants can be used to show nonclassicality [16–18], can be estimated with few measurements and have known uncertainties, a critical requirement for proofs of nonclassicality.

Approach.—Quantification or testing of distributions has features not encountered in quantification of observables. For example, experimental measurement noise appears as a distortion of the distribution that cannot be "averaged away" by additional measurements. As will be discussed later, the theory of cumulants naturally handles this situation. We focus on the fourth-order cumulant κ_4, the lowest-order indicator of non-Gaussianity in symmetric distributions such as Fock [19] and "Schrödinger kitten" states [7,11]. We study theoretically and experimentally the noise properties of Fisher's unbiased estimator of κ_4, i.e., the fourth "k statistic" k_4, and find optimal measurement conditions. Because κ_4 is related to the negativity of the Wigner function [16], this estimation is of direct relevance to detection of nonclassical states. We employ quantum nondemolition measurement, a key technique for generation and measurement of nonclassical states in atomic spin ensembles [20,21] and nanomechanical oscillators [22].

Moments, cumulants, and estimators.—A continuous random variable X with probability distribution function $P(X)$ is completely characterized by its moments $\mu_k \equiv \int X^k P(X) dX$ or cumulants $\kappa_n = \mu_n - \sum_{k=1}^{n-1} \binom{n-1}{k-1} \mu_{n-k}\kappa_k$, where $\binom{n}{k}$ is the binomial coefficient.

Since Gaussian distributions have $\kappa_{n>2} = 0$, estimation of κ_4, (or κ_3 for nonsymmetric distributions), is a natural test for non-Gaussianity. In an experiment, a finite sample $\{X_1 \dots X_N\}$ from P is used to estimate the κ's. Fisher's unbiased estimators, known as "k statistics" k_n, give the correct expectation values $\langle k_n \rangle = \kappa_n$ for finite N [23]. Defining $S_n = \sum_i X_i^n$ we have

$$k_3 = (2S_1^3 - 3NS_1S_2 + N^2S_3)/N_{(2)}, \quad (1)$$

$$k_4 = (-6S_1^4 + 12NS_1^2S_2 - 3N(N-1)S_2^2 - 4N(N-1)S_1S_3 + N^2(N+1)S_4)/N_{(3)}, \quad (2)$$

where $N_{(m)} \equiv N(N-1)\dots(N-m)$.

We need the uncertainty in the cumulant estimation to test for non-Gaussianity, or to compare non-Gaussianity between distributions. For hypothesis testing and maximum-likelihood approaches, we need the variances of k_3, k_4 for a given P. These are found by combinatorial methods and given in Ref. [23]:

$$\mathrm{var}(k_3) = \kappa_6/N + 9N(\kappa_2\kappa_4 + \kappa_3^2)/N_{(1)} + 6N^2\kappa_2^3/N_{(2)}, \quad (3)$$

$$\mathrm{var}(k_4) = \kappa_8/N + 2N(8\kappa_6\kappa_2 + 24\kappa_5\kappa_3 + 17\kappa_4^2)/N_{(1)} + 72N^2(\kappa_4\kappa_2^2 + 2\kappa_3^2\kappa_2)/N_{(2)} + 24N^2(N+1)\kappa_2^4/N_{(3)}. \quad (4)$$

It is also possible to estimate the uncertainty in k_4 from data $\{X\}$ using estimators of higher order cumulants [23]. The efficiency of cumulant estimation is illustrated in Fig. 1.

Measurement noise.—When the measured signal is $Z = X + Y$, where X is the true value and Y is uncorrelated noise, the measured distribution is the convolution $P(Z) = P(X) \otimes P(Y)$. The effect of this distortion on cumulants is the following: for independent variables, cumulants accumulate (i.e., add) [23], so that $\kappa_n^{(Z)} = \kappa_n^{(X)} + \kappa_n^{(Y)}$, where $\kappa_n^{(Q)}$, $k_n^{(Q)}$ indicate κ_n, k_n for distribution $P(Q)$. The extremely important case of uncorrelated, zero-mean Gaussian noise, $\kappa_2^{(Y)} = \sigma_Y^2$ and other cumulants zero, is thus very simple: $\kappa_n^{(Z)} = \kappa_n^{(X)}$ except for $\kappa_2^{(Z)} = \kappa_2^{(X)} + \sigma_Y^2$. Critically, added Gaussian noise does not alter the observed κ_3, κ_4.

Experimental system and state preparation.—We test this approach in a highly realistic experiment in order to understand the role of experimental imperfections by estimating classical non-Gaussian spin distributions in an atomic ensemble, similar to ensemble systems being developed for quantum networking with non-Gaussian states [24]. The collective spin component F_z is measured by Faraday rotation using optical pulses (where z is the optical propagation axis). The detected Stokes operator is $S_y^{(\text{out})} = S_y^{(\text{in})} + GN_L F_z/2$, where G is a coupling constant, N_L is the number of photons, and $S_y^{(\text{in})}$ is the input Stokes operator, which contributes quantum noise. In the above formulation $X = F_z$, $Y = 2S_y^{(\text{in})}/(GN_L)$ and $Z = 2S_y^{(\text{out})}/(GN_L)$.

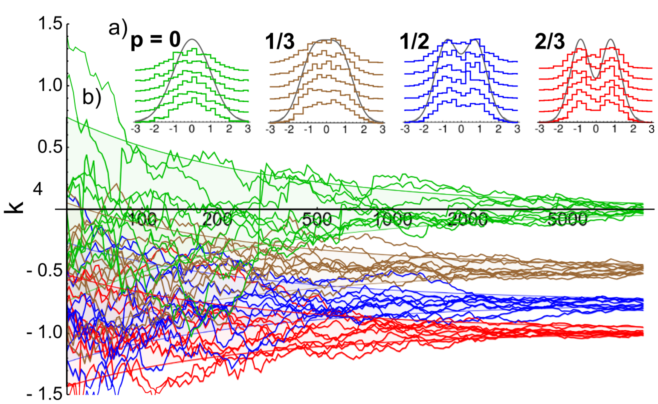

FIG. 1 (color online). Simulated estimator k_4 as a function of sample size N. (a) (insets) black curves show quadrature distributions of states $\rho = (1-p)|0\rangle\langle 0| + p|1\rangle\langle 1|$, scaled to unit variance, and six $N = 1000$ histograms (offset for clarity) for $p = 0$ (green), 1/3 (brown), 1/2 (blue) and 2/3 (red). (b) Ten realizations of k_4 versus N drawn from each of the four distributions. Shaded regions show $\kappa_4 \pm \sqrt{\text{var}(k_4)}$, from Eqs. (2) and (4). With $N = 1000$, k_4 distinguishes $p = 1/2$ (blue) from $p = 0$ (green, Gaussian) with $>7\sigma$ significance, even though the histograms look similar "to the eye." This shows the efficiency relative to histogram-based detection [15].

The experimental system is described in detail in references [21,25,26]. An ensemble of $\sim 10^6$ ^{87}Rb atoms is trapped in an elongated dipole trap made from a weakly focused 1030 nm beam and cooled to 25 μK. A nondestructive measurement of the atomic state is made using pulses of linearly polarized light detuned 800 MHz to the red of the $F = 1 \rightarrow F' = 0$ transition of the D_2 line and sent through the atoms in a beam matched to the transverse cloud size. The pulses are of 1 μs duration, contain 3.7×10^6 photons on average, and are spaced by 10 μs to allow individual detection. The 240:1 aspect ratio of the atomic cloud creates a strong paramagnetic Faraday interaction with measured coupling $G \approx 6 \times 10^{-8}$ rad/spin. After interaction with the atoms, $S_y^{(\text{out})}$ is detected with a shot-noise-limited (SNL) balanced polarimeter in the $\pm 45°$ basis. N_L is measured with a beam-splitter and reference detector before the atoms. The probing-plus-detection system is shot-noise-limited above 3×10^5 photons/pulse. Previous work with this system has demonstrated QND measurement of the collective spin F_z with an uncertainty of ~ 500 spins [21,26].

We generate Gaussian and non-Gaussian distributions with the following strategy: we prepare a "thermal state" (TS), an equal mixture of the $F = 1$, $m_F = -1,0,1$ ground states, by repeated unpolarized optical pumping between the $F = 1$ and $F = 2$ hyperfine levels, finishing in $F = 1$ [26]. By the central limit theorem, the TS of 10^6 atoms is nearly Gaussian with $\langle F_z \rangle = 0$ and $\text{var}(F_z) = \sigma^2 = 2N_A/3$. By optical pumping with pulses of circularly polarized light we displace this to $\langle F_z \rangle = \alpha$, with negligible change in $\text{var}(F_z)$ [27], to produce $P_\alpha(F_z) = (\sigma\sqrt{2\pi})^{-1} \times \exp[-(F_z - \alpha)^2/(2\sigma^2)]$. By displacing different TS alternately to α_+ and α_-, we produce an equal statistical mixture of the two displaced states, $P_\alpha^{(\text{NG})}(F_z) = [P_{\alpha_+}(F_z) + P_{\alpha_-}(F_z)]/2$. With properly chosen $\alpha_\pm$, $P_\alpha^{(\text{NG})}(F_z)$ closely approximates marginal distributions of mixtures of $n = 0,1$ Fock states and $m = N, N-1$ symmetric Dicke states. The experimental sequence is shown in Fig. 2.

Detection, analysis, and results.—For each preparation, 100 measurements of F_z are made, with readings (i.e., estimated F_z values by numerical integration of the measured signal) $m_i = 2S_y^{(\text{out},i)}/N_L^{(i)}$. Because the measurement is nondestructive and shot-noise limited, we can combine N_R readings in a "meta pulse", i.e., a train of individual pulses, with reading $M \equiv \sum m_i$. Varying the number of individual pulses combined in this way, we vary the total number of photons and thus the sensitivity of the meta-pulse, while preserving the quantum noise features [26]. These readings have the distribution $P_{\alpha_\pm}(M) = \exp[-(M - \alpha_\pm)^2/(2\sigma_M^2)]/(\sigma_M\sqrt{2\pi})$ where the variance $\sigma_M^2 = \sigma_A^2 N_A'^2 N_R^2 + \sigma_R^2$ includes atomic noise $\sigma_A^2 N_A'^2$ and readout noise, $\sigma_R^2 = N_R/N_L$ with $N_A' = N_A/N_A^{\text{MAX}}$. The variance σ_A^2 is determined from the scaling of $\text{var}(M)$ with

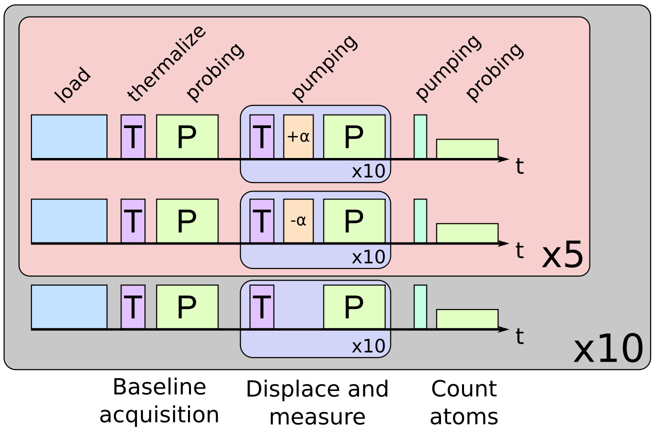

FIG. 2 (color online). Experimental sequence: The experimental sequence divides into distinct tasks. Baseline acquisition: prepare the thermal state and probe to measure the residual rotation. Displace and measure (DM[α]): prepare the thermal state, displace by α and probe. Thanks to atom loss at each thermalization, the atom number is varied by repeating DM several times. Measure number of atoms N_A: by pumping the atoms into $F = 1$, $m_F = 1$ and probing we measure the number of atoms in the trap. To correct for drifts, a sequence without displacement (DM[0]) is performed every 11 runs. We perform the sequence varying the displacement to acquire a dataset of quantum-noise-limited measurements of $P_\alpha^{(\mathrm{NG})}(S_y^{(\mathrm{out})})$ for different α. The duration of a single displace and measure event is about 1 ms, comparable to quantum memory storage times [28], and orders of magnitude longer than the ns or ps time-scales typical of optical quantum state preparation [6,15].

N_A and N_R, as in [26]. The readout noise can be varied over 2 orders of magnitude by appropriate choice of N_R. For one probe pulse and the maximum number of atoms we have $\sigma_R^2/\sigma_A^2 = 84.7$.

To produce a non-Gaussian distribution, we compose metapulses from N_R samples drawn from displaced thermal state (DM[α_+] or DM[α_-]) preparations with equal probability, giving distribution $P_\alpha^{(\mathrm{NG})}(M) = [P_{\alpha_+}(M) + P_{\alpha_-}(M)]/2$. With $\alpha_M \equiv (\alpha_+ - \alpha_-)/2$, the distribution has $\kappa_{2n+1} = 0$, $\kappa_2 = \alpha_M^2 + \sigma_M^2$, $\kappa_4 = -2\alpha_M^4$, $\kappa_6 = 16\alpha_M^6$, $\kappa_8 = -272\alpha_M^8$. Our ability to measure the non-Gaussianity is determined by $\langle k_4\rangle = \kappa_4$ and from Eq. (4)

$$\mathrm{var}(k_4) = 136N\alpha_M^8/N_{(1)} - 144N^2\alpha_M^4(\alpha_M^2 + \sigma_M^2)^2/N_{(2)} + 24N^2(N+1)(\alpha_M^2 + \sigma_M^2)^4/N_{(3)}. \quad (5)$$

As shown in Fig. 3, the experimentally obtained values agree well with theory, and confirm the independence from measurement noise.

The "signal-to-noise ratio" for κ_4, $S = {\kappa_4}^2/\mathrm{var}(k_4)$, is computed using Eq. (5), $\kappa_4 = -2\alpha_M^4$, and experimental α_M, N_R, σ_R, is shown as curves in Fig. 4. We can confirm this S experimentally by computing $S_N \equiv \langle k_4\rangle^2/\mathrm{var}(k_4)$ using k_4 values derived from several realizations of the experiment, each sampling P_α^{NG} N times. In the limit of many realizations $S_N \to S$. We employ a bootstrapping technique: From 100 samples of $P_\alpha^{(\mathrm{NG})}(M)$ for given parameters α_M, N_R and N_A, we derive 33 $N = 20$ realizations by random sampling without replacement, and compute $\langle k_4\rangle$ and $\mathrm{var}(k_4)$ on the realizations. As shown in Fig. 4, reasonable agreement with theory is observed over a wide range of parameters. Without evaluating still higher order statistics it is difficult to say if the remaining differences are statistical or systematic.

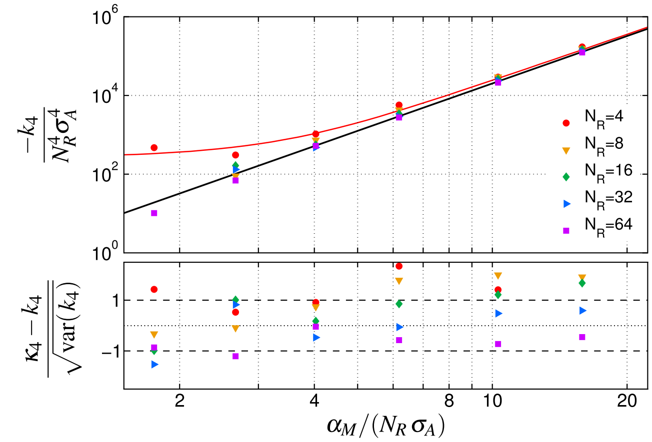

FIG. 3 (color online). Measured and predicted k_4 with residuals for non-Gaussian distributions of different α. Readout noise is varied by the choice of N_R. Data is normalized to N_R and σ_A. Top: Points show normalized $-k_4$ calculated from $N = 100$ preparations of the ensemble with different α (horizontal axis), and N_R (colors). Black line indicates expected $-\kappa_4$, red line (top) shows $-\kappa_4 + \sqrt{\mathrm{var}(k_4)}$ calculated from the distribution parameters for the largest readout noise. Some points have negative values and are not shown because of the logarithmic scale. Bottom: normalized residuals $(-k4 + \kappa_4)/\sqrt{\mathrm{var}(k_4)}$. The normalization is done with the expected $\mathrm{var}(k_4)$ for each N_R. Measured k_4 agrees well with theory, in particular, measurement noise increases the observed variance, but not the expectation.

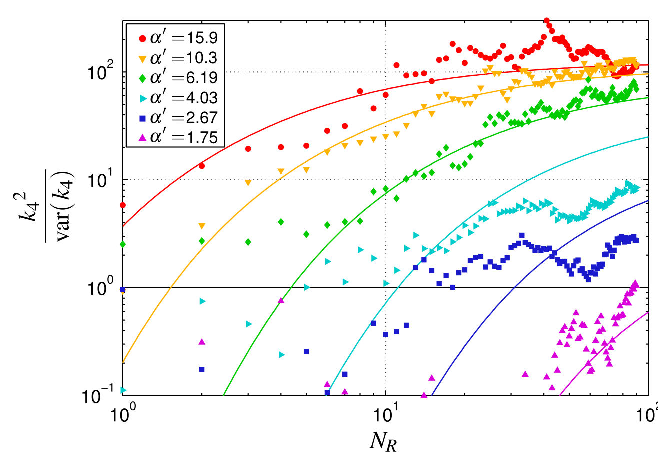

FIG. 4 (color online). Signal-to-noise in estimation of κ_4 versus readout noise for different $\alpha' = \alpha_M/(N_R\sigma_A)$. Points show measurement results, lines show theory. (details in the text)

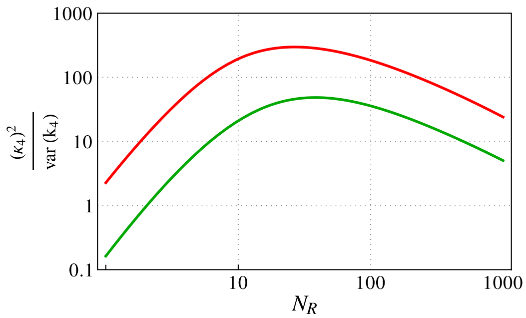

FIG. 5 (color online). Signal-to noise-ratio S versus N_R for a fixed probe number $N_M N_R = 1 \times 10^5$ for the probability distribution associated with Fock state mixture described in the text with a normalized $n = 0$ width $\sigma_0 = 1$. Red curve (top): $p = 1$. Green curve (bottom): $p = 0.5$ with SNL measurement: $\sigma_R = \sqrt{20/N_R}$.

Optimum estimation of non-Gaussian distributions.—Finally, we note that in scenarios where measurements are expensive relative to state preparation (as might be the case for QND measurements of optical fields or for testing the successful storage of a single photon in a quantum memory), optimal use of measurement resources (e.g. measurement time) avoids both too few preparations and too few probings.

We consider a scenario of practical interest for quantum networking: a heralded single-photon state is produced and stored in an atomic ensemble quantum memory. Assuming the ensemble is initially polarized in the $\hat{X}$ direction, the storage process maps the quadrature components X, P onto the corresponding atomic spin operators X_A, $P_A \propto F_z$, $-F_y$, respectively. QND measurements of F_z are used to estimate X_A, and thus the non-Gaussianity of the stored single photon. Because of imperfect storage, this will have the distribution of a mixture of $n = 0$ and $n = 1$ Fock states: $\rho = (1-p)|0\rangle\langle 0| + p|1\rangle\langle 1|$. For a quadrature X, we have the following probability distribution $P_p(X) = \exp[-x^2/(2\sigma_0^2)](px^2/\sigma_0^2 + 1 - p)/(\sqrt{2\pi}\sigma_0)$, where σ_0 is the width of the $n = 0$ state.

Taking in account the readout noise σ_R^2, the cumulants are $\kappa_{\mathrm{odd}} = 0$, $\kappa_2 = (2p+1)\sigma_H^2 + \sigma_R^2$, $\kappa_4 = -12p^2\sigma_H^4$, $\kappa_6 = 240p^3\sigma_H^6$, $\kappa_8 = -10\,080p^4\sigma_H^8$, where the readout noise σ_R^2 is included as above. Here, κ_4 is directly related to the classicality of the state, since $p > 0.5$ implies a negative Wigner distribution [19].

For a fixed total number of measurement resources $N_M N_R$, an optimal distribution of resources per measurement N_R exists as shown in Fig. 5. With increasing N_R, the signal-to-noise first increases due to the improvement of the measurement precision. Then, once the increased measurement precision no longer gives extra information about k_4, the precision decreases due to reduced statistics because of the limited total number of probes. For a large total number of measurements, we can derive a simplified expression of this optimum. We derive asymptotic expressions of S: S_L (S_H) for $\sigma_R \ll \sigma_0$ ($\sigma_R \gg \sigma_0$). The optimal N_R is found by solving $S_L = S_R$ giving $\sigma_R^8 \approx \sigma_0^8(1 + 8p - 12p^2 + 48p^3 - 24p^4)$. For this optimal σ_R, the measurement noise is in the same order of magnitude as the characteristic width of the non-Gaussian distribution.

Conclusion.—The cumulant-based methods described here should be very attractive for experiments with non-Gaussian states of material systems such as atomic ensembles and nanoresonators, for which the state preparation time is intrinsically longer, and for which measurement noise is a greater challenge than in optical systems. Cumulant-based estimation is simultaneously efficient, requiring few preparations and measurements, accommodates measurement noise in a natural way, and facilitates statistically-meaningful tests, e.g., of nonclassicality. Experimental tests with a cold atomic ensemble demonstrate the method in a system highly suitable for quantum networking, while the theory applies equally to other quantum systems of current interest.

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